CLASA a XI a 2020

Să se calculeze : $\lim\limits_{n\to+\infty}\left(\lim\limits_{x\to 0}\left(\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}\right)^{\frac{1}{x^2}}\right)$, $n\in\mathbf{N}^*$.

REZOLVARE

2p Fie $L=\lim\limits_{x\to 0}\left(\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}\right)^{\frac{1}{x^2}}$; adunăm si scădem 1 obținem $L=e^{\lim\limits_{x\to 0}\frac{\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}-1}{x^2}}$;

2p $\lim\limits_{x\to 0}\frac{\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}-1}{x^2}=$

$$\lim_{x\to 0}\frac{e^{\ln\left(\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}\right)}-1}{\ln\left(\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}\right)}\cdot\lim_{x\to 0}\frac{\ln\left(\sqrt{\frac{x^2}{3}+1}\cdot\sqrt[3]{\frac{x^2}{4}+1}\cdot\ldots\cdot\sqrt[n]{\frac{x^2}{n+1}+1}\right)}{x^2}=$$

2p $=\lim\limits_{x\to 0}\left(\frac{1}{2}\cdot\frac{\ln\left(\frac{x^2}{3}+1\right)}{x^2}+\frac{1}{3}\cdot\frac{\ln\left(\frac{x^2}{4}+1\right)}{x^2}+\ldots+\frac{1}{n+1}\cdot\frac{\ln\left(\frac{x^2}{n}+1\right)}{x^2}\right)=\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}$;

1p avem: $\lim\limits_{n\to+\infty}L=\lim\limits_{n\to+\infty}e^{\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}}=\sqrt{e}$

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Se considera matricea $A \in M_3(R)$ astfel incat $A\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 2 & 9 & 5 \\ 2 & 3 & 25 \end{pmatrix}$.

a. Sa se determine matricea A;

b. Sa se calculeze A^n, unde $n \in N^*$.

Solutie.

$a)$Din calcul rezulta $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}^{-1} = \frac{1}{22}\begin{pmatrix} 14 & -4 & -2 \\ -4 & 9 & -1 \\ -2 & -1 & 5 \end{pmatrix}$ (1), *deci* $A = \begin{pmatrix} 4 & 3 & 5 \\ 2 & 9 & 5 \\ 2 & 3 & 25 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}^{-1} =$ **3p**

$\frac{1}{22}\begin{pmatrix} 4 & 3 & 5 \\ 2 & 9 & 5 \\ 2 & 3 & 25 \end{pmatrix}\begin{pmatrix} 14 & -4 & -2 \\ -4 & 9 & -1 \\ -2 & -1 & 5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 17 & 3 & 7 \\ -9 & 34 & 6 \\ -17 & -3 & 59 \end{pmatrix}$;

$b)$Din ipoteza avem $\begin{cases} A\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ A\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \\ A\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \end{cases} \overset{inductie}{\Rightarrow} \begin{cases} A^n\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 2^n \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ A^n\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = 3^n \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \\ A^n\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = 5^n \cdot \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \end{cases} \Rightarrow A^n\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 2\cdot 2^n & 3^n & 5^n \\ 2^n & 3\cdot 3^n & 5^n \\ 2^n & 3^n & 5\cdot 5^n \end{pmatrix} \Rightarrow$

$A^n = \begin{pmatrix} 2\cdot 2^n & 3^n & 5^n \\ 2^n & 3\cdot 3^n & 5^n \\ 2^n & 3^n & 5\cdot 5^n \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix}^{-1} \overset{(1)}{=} \frac{1}{22}\begin{pmatrix} 2\cdot 2^n & 3^n & 5^n \\ 2^n & 3\cdot 3^n & 5^n \\ 2^n & 3^n & 5\cdot 5^n \end{pmatrix}\begin{pmatrix} 14 & -4 & -2 \\ -4 & 9 & -1 \\ -2 & -1 & 5 \end{pmatrix} =$ **4p**

$\frac{1}{22}\begin{pmatrix} 28\cdot 2^n - 4\cdot 3^n - 2\cdot 5^n & -8\cdot 2^n + 9\cdot 3^n - 5^n & -4\cdot 2^n - 3^n + 5\cdot 5^n \\ 14\cdot 2^n - 12\cdot 3^n - 2\cdot 5^n & -4\cdot 2^n + 27\cdot 3^n - 5^n & -2\cdot 2^n - 3\cdot 3^n + 5\cdot 5^n \\ 14\cdot 2^n - 4\cdot 3^n - 10\cdot 5^n & -4\cdot 2^n + 9\cdot 3^n - 5\cdot 5^n & -2\cdot 2^n - 3^n + 25\cdot 5^n \end{pmatrix}$.

Soluţie : $x_{n+1}-\frac{\pi}{4}=\sin x_n-\cos x_n=\sqrt{2}\sin(x_n-\frac{\pi}{4})$ **1p**

Notăm $y_n=x_n-\frac{\pi}{4}\ \Rightarrow\ y_1=-\frac{\pi}{4}$ si

$y_{n+1}=\sqrt{2}\sin y_n,\forall n\geq 1$. Fie $f:R\to R$, $f(x)=\sqrt{2}\cdot\sin x$, strict crescătoare pe $[-\frac{\pi}{2},0]$. **......1p**

Cum $y_1=-\frac{\pi}{4}\in[-\frac{\pi}{2},0]$, presupunem că $-\frac{\pi}{2}\leq y_n\leq 0\ \Rightarrow\ f(-\frac{\pi}{2})\leq f(y_n)\leq f(0)\ \Leftrightarrow$

$\Leftrightarrow\ -\sqrt{2}\leq y_{n+1}\leq 0\ \Rightarrow\ -\frac{\pi}{2}\leq-\sqrt{2}\leq y_{n+1}\leq 0$. Deci $y_n\in[-\frac{\pi}{2},0]$, $\forall n\geq 1$. **......1p**

$y_2=\sqrt{2}\sin y_1=\sqrt{2}\cdot(-\frac{\sqrt{2}}{2})=-1\ \Rightarrow\ y_2<y_1=-\frac{\pi}{4}$. Presupunem că $y_{n+1}<y_n\ \Rightarrow\ \Rightarrow$ $f(y_{n+1})<f(y_n)\ \Leftrightarrow\ y_{n+2}<y_{n+1}$. Rezultă că $(y_n)_{n\geq1}$ este strict descrescător $\Rightarrow$ $(y_n)_{n\geq1}$ convergent $\Rightarrow$ $(x_n)_{n\geq1}$ convergent . **......1p**

Notăm $l=\lim\limits_{n\to\infty}y_n\in[-\frac{\pi}{2},0)$. Trecem la limită în relaţia de recurenţă $\Rightarrow l=\sqrt{2}\cdot\sin l\Rightarrow$

$\Rightarrow\ \frac{l}{\sqrt{2}}=\sin l$. **......1p**

Fie funcţiile $g_1,g_2:[-\frac{\pi}{2},0]\to R$, $g_1(x)=\frac{x}{\sqrt{2}}$, $g_2(x)=\sin x$, funcţii continue şi strict

crescătoare, g_1 liniară, g_2 convexa , $g_1(-\frac{\pi}{2})<g_2(-\frac{\pi}{2})$ si $g_1(0)=g_2(0)$

$\Rightarrow$ ecuaţia $g_1(l)=g_2(l)$ are soluţie unică în $(-\frac{\pi}{2},0)$. **......1p**

Cum $g_1(-\frac{\pi}{2})<g_2(-\frac{\pi}{2})$ şi $g_1(-\frac{5\pi}{12})>g_2(-\frac{5\pi}{12})$ ($\Leftrightarrow -\frac{5\pi}{12\sqrt{2}}>\sin(-\frac{5\pi}{12})\ \Leftrightarrow$

$-\frac{5\pi}{12\sqrt{2}}>-\frac{\sqrt{6}+\sqrt{2}}{4}\ \Leftrightarrow\ \Leftrightarrow 5\pi<6+6\sqrt{3}$, adevărat), ecuaţia $g_1(l)=g_2(l)$ are soluţia în

$(-\frac{\pi}{2},-\frac{5\pi}{12})\ \Rightarrow \lim\limits_{n\to\infty}x_n=l+\frac{\pi}{4}\in(-\frac{\pi}{4},-\frac{\pi}{6})$. **......1p**

XI

Se dă şirul $(a_n)_{n\geq 1}$, $a_1=\alpha\in(0,1)$ şi $a_{n+1}=a_n^2-a_n+1$ $(\forall)\ n\geq 1$.

Definim şirul $(b_n)_{n\geq 1}$ prin: $b_n=\begin{vmatrix}1 & 1 & 1\\ a_n & a_{n+1} & a_{n+2}\\ a_{n+3} & a_{n+4} & a_{n+5}\end{vmatrix}$ $\forall n\geq 1$.

Calculaţi $\lim\limits_{n\to\infty} n^k\cdot b_n$, $k\in\mathbb{N}^*$.

EMIL VASILE

S: (a_n) este crescător şi are limita 1. $a_{n+1}-1=a_n(a_n-1)$

$a_{n+1}-a_n=(a_n-1)^2$ $\forall n$. Inductiv $a_n<1$. Notăm $\lim a_n=x \Rightarrow$
$x^2-x+1=x\Rightarrow x=1$.

$$b_n=\begin{vmatrix}a_{n+1}-a_n & a_{n+2}-a_{n+1}\\ a_{n+4}-a_{n+3} & a_{n+5}-a_{n+4}\end{vmatrix}=\begin{vmatrix}(a_n-1)^2 & (a_{n+1}-1)^2\\ (a_{n+3}-1)^2 & (a_{n+4}-1)^2\end{vmatrix}=$$

$$=(a_n-1)^2(a_{n+3}-1)^2\begin{vmatrix}1 & a_n^2\\ 1 & a_{n+3}^2\end{vmatrix}=(a_n-1)^2\cdot(a_{n+3}-1)^2\cdot(a_{n+3}-a_n)\cdot(a_{n+3}+a_n) \quad (1)$$

Dar $a_{n+3}-a_n=a_{n+3}-a_{n+2}+a_{n+2}-a_{n+1}+a_{n+1}-a_n=(a_{n+2}-1)^2+(a_{n+1}-1)^2+(a_n-1)^2$.

Deci $b_n=(a_n-1)^2(a_{n+3}-1)^2\left[(a_n-1)^2+(a_{n+1}-1)^2+(a_{n+2}-1)^2\right](a_n+a_{n+3})$ (2)

Calculez $\lim\limits_n n(1-a_n)=\lim \dfrac{n}{\frac{1}{1-a_n}}\overset{C.S.}{=}\lim\dfrac{(1-a_n)(1-a_{n+1})}{a_{n+1}-a_n}=\lim\dfrac{1-a_{n+1}}{1-a_n}=1$ (2)

$\infty\leftarrow\dfrac{1}{1-a_n}$

Pentru $k=6$ $\lim\limits_n n^6 b_n=\lim\limits_n\left[n(a_n-1)\right]^2\cdot\left[(n+3)(a_{n+3}-1)\right]^2(a_n+a_{n+3})\cdot$

$$\left[(n(a_n-1))^2+\frac{n^2}{(n+1)^2}\left((n+1)(a_{n+1}-1)\right)^2+\frac{n^2}{(n+2)^2}\left((n+2)(a_{n+2}-1)\right)^2\right]\cdot\frac{n^2}{(n+3)^2}=6$$

Evident $k>6\Rightarrow \lim n^k b_n=\infty$

$k<6$ $\lim n^k b_n=0$